FLOW OF AN IDEAL FLUID

1.1 Introduction

Hydraulic and aerodynamic engineering analysis and design involve predictions of patterns of fluid motion and of fluid forces associated with those patterns. The selection of the appropriate method for making the predications is determined by the nature of the particular problem and by the precision desired. In some cases the task is quite simple, as, for example, the precise determination, by the principles of hydrostatics, of pressure forces exerted by a stationary body of water. Certain cases of viscous flow can be solved, either precisely or with negligible error, by means of the viscous flow equations due to Navier and Stokes. In the great majority of fluid flow problems, however, precise analytical determinations of forces and velocities are not possible, owing to the complex effects upon the flow of fluid viscosity. In such cases, recourse must be had to simplifying assumptions in order that approximate analytical solutions may be obtained.

The simplest and most common approximation is the method of analysis in one dimension, which yields entirely adequate solutions to many problems requiring the determination of total forces rather than pressure distributions, or of average velocities rather than velocity distributions. In a one-dimensional analysis, velocities and pressures are assumed to vary with distance only in the general direction of flow, and mean values of the velocity on planes normal to this flow direction are adopted for purposes of calculation. On these planes, velocity variations due to the effects of viscosity and to changes in boundary alignment are ignored *(Fig. 1.1).*

Pressure and velocity distributions in general cannot be determined by means of the one-dimensional approach. In certain cases, they can be determined with good precision by analysis in two or three dimensions and, where such methods are inadeq1uate, laboratory model tests must be undertaken.

Most analytical approaches are based upon the methods of classical hydrodynamics, the branch of applied mathematics which treats of the perfect or ideal fluid. This hypothetical fluid is, by definition, incompressible and non-viscous, so that it experiences no shearing stresses and its elements in contact with solid boundaries slip tangentially, without resistance, along those boundaries. The fluid force on any element of the boundary surface is normal to that surface.

In the case of motion of a real fluid, shearing stresses are always present and there is no slip at the boundary. The fluid particles in contact with the boundary adhere to it and have no tangential motion along it. These fundamental differences between ideal and real fluids with respect to slip and shear stresses account for the divergence of theoretical predictions from experimental fact in many case of real fluid flow. According to the ordinary theory of ideal fluid flow, for example, a body which moves through a fluid of infinite extent experiences no drag force (d’ Alembert’s paradox). In a real fluid, the condition of no slip between the surface of the body and the fluid elements in contact with it, and existence of viscous tangential stresses in the neighbouring fluid layers together account for the ever-present drag on the moving body.

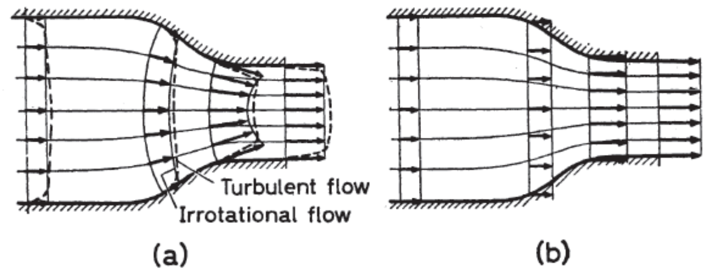

Figure 1.1—Velocity distribution in a two-dimensional contraction. (*a*) Irrotational flow and turbulent flow, (*b*) one-dimensional approximation

There are, nevertheless, many practical problems in fluid dynamics which can be solved with fair precision by the methods of classical hydrodynamics, in particular, by the potential flow theory. They include such cases as the distributions of velocity and pressure around the leading portions of streamlined objects and in flow in converging passages, the form and motion of surface waves on a liquid, the profiles of free jets and weir nappies, pressure distributions resulting from impulsive actions (prior to the occurrence of appreciable fluid motion) and, paradoxically for an ideal fluid theory, certain classes of essentially viscous motion, such as percolation through granular materials.

A sound appreciation of the possibilities and limitations of classical hydrodynamic theory as a practical engineering design tool requires a knowledge of elementary hydraulics, which it is assumed the reader possesses, and some familiarity with modern developments in the science of the mechanics of fluids. Therefore, following the treatment of the fundamentals of ideal fluid flow in this chapter, attention is directed, in the next, to the characteristics of real fluid flow which distinguish it to a greater or lesser extent from ideal fluid flow. The remainder of the books is devoted to patterns of ideal fluid flow and methods of predicting those patterns, and it the mind, so that the possibilities and limitations of ideal flow theory as a practical design approach may be assessed correctly.

1.2 Fluid properties

A fluid is a material which flows. Its capacity to flow, which distinguishes it from a solid, results from the fact that its particles can be readily displaced under the action of shearing stresses. Like solids, the fluids known as the liquids offer considerable resistance to compression and tensile stresses but all fluids deform readily and continuously under the action of a shearing stress, so long as that stress operates. If the rate of shear deformation is small, the fluid offers negligible resistance, which disappears, however, once the deforming motion ceases. The resistance arises from the existence of viscosity in the fluid.

A Newtonian fluid is one in which the shearing stress, in one directional flow, is proportional to the rate of deformation as measured by the velocity gradient across the flow. Thus,

$$\tau = \mu \frac{du}{dy} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.1)$$

where τ is the shearing stress in the x-direction on planes normal to the y-axis; u is the velocity in the x-direction, varying only with y; and μ, the constant of proportionality, is the dynamic viscosity or coefficient of viscosity, which for any particular fluid varies significantly only with change in temperature *(Fig. 1.2).* The common fluids, such as air, water and light petroleum oils, are Newtonian fluids. Non-Newtonian fluids, whose behavior does not conform to Eq. 1.1, are not usually net in engineering practice and will not be considered further.

A liquid is a fluid which, at a given temperature, occupies a definite volume which is little affected by change in pressure. Poured into a stationary container, the liquid will occupy the lower part of the container and form a free level surface in most hydraulic calculations, liquids can regarded as incompressible. In fact, under many conditions, it is convenient and reasonable to regard even a gas such as atmospheric air as incompressible, provided that the velocities involved are small compared with the velocity of sound in the gas.

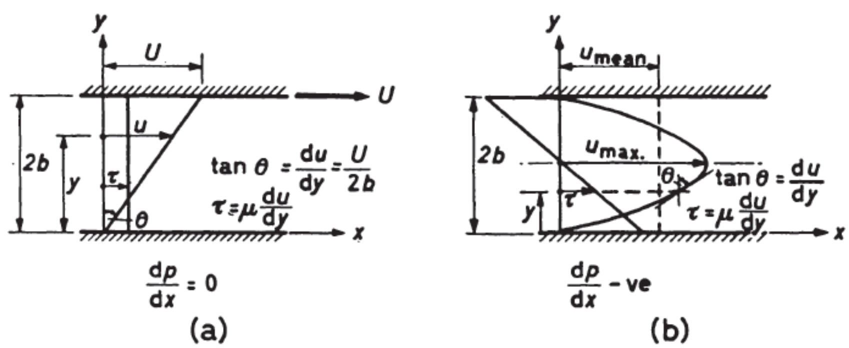

Figure 1.2—Relationship between shear stress and velocity gradient in one-directional flow between parallel plates. (*a*) One plate moving in its own plane, zero pressure gradient, (*b*) both plates fixed, negative pressure gradient

The mathematical approach to study of fluid flow is greatly simplified if attention is restricted to a hypothetical "ideal", or "perfect" fluid, which is assumed to possess zero viscosity. In common with real fluids, it has density and it flows, but it offers no resistance to shearing deformations so that there can be no shear stress in an ideal fluid. It is assumed to exhibit no surface tension effects and it does not vaporize, so that cavitation, or the formation of low pressure vapour pockets, does not occur. The density of the fluid and the pressure, velocity and acceleration, if they vary with position, are assumed to vary continuously from point to point, except possibly at isolated points, lines or surfaces of discontinuity, which are called "singularities".

The characteristics of flow of an ideal fluid in many cases resemble those of real fluid flow and the simplified mathematical approach yields useful practical results. In this chapter the fundamentals of ideal fluid flow are investigated. In the next chapter attention is directed to the modifying effects of viscosity and the resulting limitations of the ideal fluid flow theory in its application to real fluid flow.

1.3 Pressure at a point

All matter is, on a very small scale, discontinuous. In order to provide unambiguous meanings to such terms as the density, or the velocity, or the pressure "at a point", it is necessary to define a “point” as a very small volume which is yet very large compared with the spacing of fluid molecules. Alternatively, the fluid is assumed to be a continuous medium in which point values correspond to the average values in the small volumes mentioned above.

The forces acting upon a small element in a mass of fluid are classified for convenience as small element in a mass of fluid are classified for convenience as body forces and surface forces. A body force is an external force, which is proportional to the volume of the element and acts “at a distance” through its centre of mass. Gravity forces are the only body forces with which the hydraulic engineer is concerned. A surface force an internal force with a magnitude proportional to the area upon which it acts. Internal pressure and viscous shear forces are the surface forces involved in fluid flow. In an ideal fluid, since there is no tangential or shearing stress, the only surface forces are pressure forces normal to the areas they act upon. The interrelationship of the stresses at a point resulting from these forces is developed for two-dimensional ideal fluid flow, with reference to *Fig. 1.3* in which for purposes of generality the *y*-axis is inclined at some angle β to the line of action of gravity. The lengths of the sides of the triangular element are respectively *a*, *b* and *c* and the corresponding normal pressures are $P_x, P_y \; and \; P_\alpha$. The fluid density is ρ and the acceleration due to gravity is *g*.

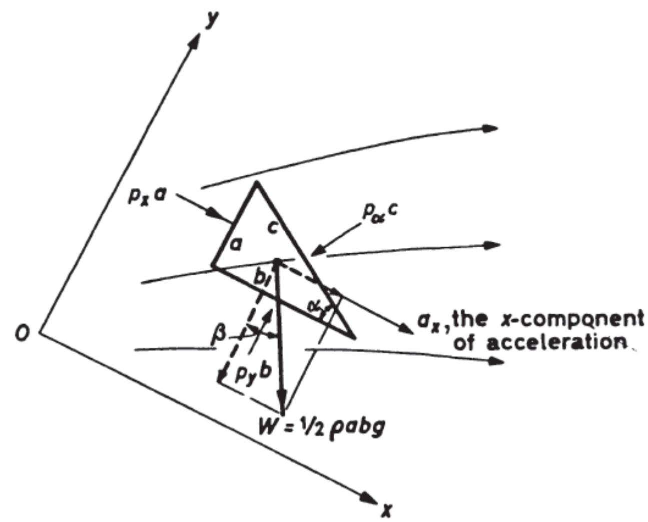

Figure 1.3—Forces acting on an element of fluid in two-dimensional non-viscous flow

The sum of the *x*-direction components of the forces acting upon the element within the fluid equals the product of the mass of the element and its acceleration, a_x, in the *x*-direction, in accordance with Newton’s second law of motion.

$$\therefore \;\; P_x a - P c \sin\alpha + \frac{1}{2}\rho abg \sin\beta = \frac{1}{2}\rho ab a_x$$

$$\therefore \;\; P_x - P_\alpha \frac{a}{b} + \frac{1}{2}\rho bg \sin\beta = \frac{1}{2}\rho b a_x$$

In order to consider the stress *at a point*, let *a* and *b* approach zero. The equation reduces to

$$P_x - P_\alpha = 0$$

Since a similar treatment of the force components acting in the *y*-direction shows that $P_y - P_\alpha = 0$, it follows that

$$P_x = P_y \qquad \dots\dots\dots\dots\dots\dots\dots (1.2a)$$

For the three-dimensional case, it can be shown that

$$P_x = P_y = P_z \qquad \dots\dots\dots\dots\dots\dots\dots (1.2b)$$

So that, in an ideal fluid, even if the fluid is accelerating, the pressure at a point is the same in all directions.

1.4 Equation of continuity

The velocity components and the density of the fluid at a point are related through the requirement that the fluid must be continuous, both in space (that is, no voids occur in the fluid) and in time (that is, fluid mass is neither created nor destroyed). The relationship holds for both viscous and non-viscous fluids. Consider an elemental parallelepiped of dimensions $\delta_x, \delta_y, \delta_z,$ through which fluid is flowing ($Fig.$ 1.4). If the centre of the elements is at *(x, y, z)* and the velocity components at the time *t* at this point are respectively *u, v* and *w,* then the mass flow rate past the centre, through the element in the *x*-direction is mass-per-unit-volume)× velocity× area, $\rho u\ \delta y\ \delta z$. The mass flow rate in through the face nearer the origin a distance $\frac{1}{2}\delta x$ is $\left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{1}{2}\delta x\right)\delta y \delta z$ and the mass flow rate out through the face farther from the origin a distance $\frac{1}{2}\delta x$ is $\left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{1}{2}\delta x\right)\delta y \delta z$. The nett gain in mass per unit time, within the element from these two faces is $-\frac{\partial(\rho u)}{\partial x}\delta x \delta y \delta z$. Similarly, the gains in mass per unit time from the other two pairs of faces are $-\frac{\partial(\rho u)}{\partial y}\delta x \delta y \delta z$ and $-\frac{\partial(\rho \omega)}{\partial z}\delta x \delta y \delta z$

The total gain in mass per unit time from all faces is $-\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho u)}{\partial y} + \frac{\partial(\rho \omega)}{\partial z}\right]\delta x \delta y \delta z$

Which must equal the time rate of increase in mass $\frac{\partial}{\partial t}(\rho \delta x \delta y dz)$ or $\frac{\partial \rho}{\partial t}\rho \delta x \delta y dz$ whence

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho u)}{\partial y} + \frac{\partial(\rho \omega)}{\partial z} = 0 \quad (1.3)$$

which is the Equation of Continuity. It is applicable throughout all fluids, except at isolated singularities.

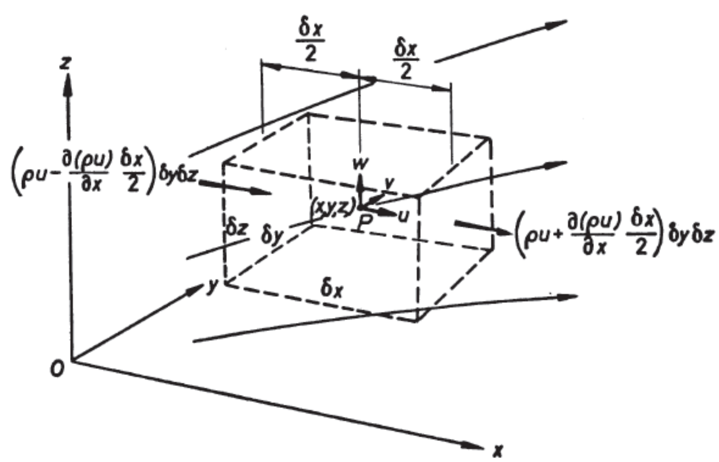

Figure 1.4—Equation of Continuity. Mass flows in the *x*-direction across the faces of a parallelepiped in three-dimensional flow

For incompressible fluids, with ρ constant, the Equation of Continuity reduces to

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad \text{... ... ...} (1.4)$$

If the velocity component is constant in one direction say, the *z*-direction, the corresponding term disappears from the continuity equation which reduces to the two-dimensional form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad \text{... ... ...} (1.5)$$

One-dimensional considerations lead to the simpler form of the continuity equation of elementary hydraulic texts. The tube-shaped volume in *Fig. 1.5* is so located in the flow that fluid flows only axially along it; flow does not occur inwards or outwards through its sides but only across its ends. If its cross-sectional area and the mean velocity of flow midway along its length are respectively *A* and *V*, both of these being functions of the distance, *s*, along the axis of the tube, an approach similar to that adopted above yields the relationship

$$\frac{\partial(\rho A)}{\partial t}+\frac{\partial(\rho AV)}{\partial s}=0$$

For incompressible fluids with ρ constarnt

$$\frac{\partial A}{\partial t}+\frac{\partial(AV)}{\partial s}=0$$

and for steady flow, $\frac{\partial A}{\partial t}=0$ hence $\frac{d(AV)}{ds}=0$

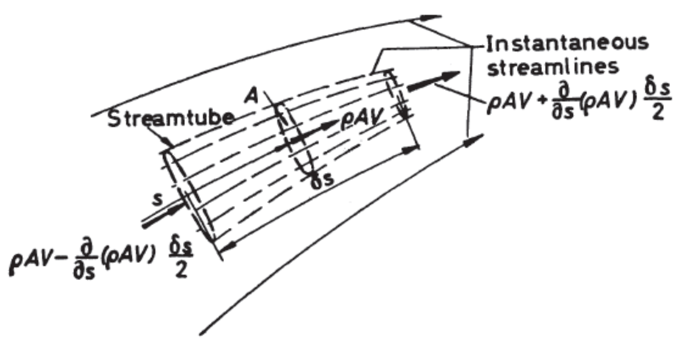

Figure 1.5—Equation of Continuity. Mass flows across the faces of a streamtube

$$\therefore \quad AV=\text{constant} \qquad \text{... ... ...} (1.6)$$

In this form the continuity equation for steady flow of an incompressible fluid relates the *mean* velocity in a given direction to the flow cross-sectional area normal to that direction. Eqs. 1.3, 1.4 and 1.5, on the other hand, show the interrelationship of the velocity components at any point in the flow.

1.5 Boundary conditions

The data necessary for an analysis of a fluid flow problem must include sufficient information concerning all of the boundaries including any arbitrarily located fluid inflow and out flow boundaries. The analysis consists of the application of the principles of fluid mechanics so as to predict the behavior of the fluid when subjected to these boundary conditions.

The possible conditions specified form boundary may include its nature, whether solid, fluid or free surface, its geometrical form, the pressure distribution on it or the velocity distribution along or across it.

An ideal fluid flowing along a solid boundary its assumed to remain in contact with the boundary without penetrating it, so that a fluid particle on a stationary. If the boundary is moving, the velocity normal to the boundary of a particle at a point on it must be equal to the velocity of the boundary normal to itself at that point. If *l, m* and *n* are the direction cosines of the inward normal to the surface at a point (that is, the cosines of the angles between the

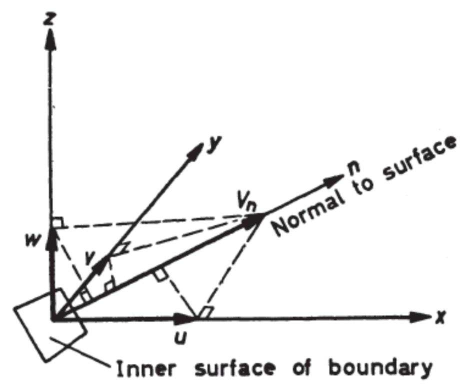

Figure 1.6—Velocity relationships at a solid boundary

particle's velocity components, *u, v* and *w,* and the normal drawn into the fluid) the components of *u, v* and *w,* along the normal are *lu, mv,* and *nw,* respectively. The particle velocity along the inward normal is the arithmetic sum of these components *(Fig. 1.6).* If the boundary velocity normal to itself at the point is V_n then

$$lu+mv+nw = V_n \qquad \text{……………………}(1.7a)$$

and if the boundary is stationary,

$$lu+mv+nw = 0 \qquad \text{……………………}(1.7b)$$

Use will be made of these equations in the consideration of energy relationships in Section 1.15.

It is evident that fluid particles on a stationary or moving boundary must remain in contact with it, their movements being wholly tangential to it. However, there may be isolated points or lines of discontinuity, to be discussed later, where the fluid particles do, in fact, leave the boundary. If the equation of the boundary surface is *F(x, y, z, t)*= 0 the co-ordinates of any fluid particle on the boundary must continusly satisfy this equation.

Suppose that, in a small element of time, δt, a particle moves along the boundary through a short distance whose components are $\delta x, \delta y, \delta z$. Since its new position must satisfy the equation of the boundary surface, the change δF must be zero.

$$\therefore \delta F = \frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial z}\delta z + \frac{\partial F}{\partial t}\delta t = 0$$

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial F}{\partial y}\frac{\delta y}{\delta t} + \frac{\partial F}{\partial z}\frac{\delta z}{\delta t} + \frac{\partial F}{\partial t} = 0$$

and, therefore, as $\delta t \to 0$

$$\frac{dF}{dt} = u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} + \frac{\partial F}{\partial t} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1.7c)}$$

where *u*, *v* and *w* are the velocity components of the particle.

This equation must be satisfied at all points on a boundary surface (except at points of discontinuity in the flow pattern).

Example

1.1 The parabolic profile $y = kx^{1/2}$ moves in the negative *x*-direction with a velocity *U*, through a fluid which was initially stationary. If *u* and *v* are the instantaneous velocity components of a fluid particle on the boundary, show that $\frac{v}{u-U} = \frac{k^2}{2y}$

Solution.—The equation of the moving profile is $y = k(x - Ut)^{1/2}$ or $y^2 = k^2(x - Ut)$ or

$$F = y^2 - k^2(x - Ut) = 0$$

with *t* measured from the instant the profile is tangential to the *y* axis (*U* being negative). Since $\frac{dF}{dt} = 0$ for all particles on the profile

$$u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + \frac{dF}{dt} = -uk^2 + 2vy + Uk^2 = 0$$

$$\therefore \quad 2vy = (u - U)k^2$$

$$\therefore \frac{v}{u-U} = \frac{k^2}{2y}$$

If the co-ordinate axes are considered to move with the profile, *U* becomes zero and the undisturbed fluid has a steady velocity of magnitude *U* in the positive *x*-direction. The profile is now tangential to the *y*-axis and, for fluid particles in contact with it, the above equation becomes

$$\frac{u}{u} = \frac{k^2}{2y}$$

that is, the slope, $\frac{u}{u'}$ of the velocity vector equals the slope of the profile, since $\frac{dy}{dx} = \frac{d}{dx}\left(kx^{1/2}\right) = \frac{k^2}{2y}$

In addition to the above kinematical condition for solid boundaries, certain other physical boundary conditions can be stated briefly. Stationary fluids and non-viscous fluids exert pressure forces which are wholly normal to the elements of solid boundaries, while forces exerted by moving viscous fluids have tangential (shearing) as well as normal components. When two different fluids are contact the pressure must be the same in each fluid at a point in the surface of contact, that is, the pressure cannot suddenly increase along a line passing through the surface. In the case of motion of a body under the action of a finite force through a fluid which extends to infinity, an essential condition is that the velocity of the fluid at infinity remain unchanged by the body's motion. Otherwise the action of a finite force would be imparting kinetic energy to an infinite mass of fluid in a finite time, which is impossible.