CHAPTER

SIX

SURFACES

6–1 INTRODUCTION

Surfaces and their description play a critical role in design and manufacturing. The design and manufacture of automobile bodies, ship hulls, aircraft fuselages and wings; propeller, turbine, compressor and fan blades; glassware and bottles; furniture, and shoes are obvious examples. Surface shape or geometry is the essence of design for either functional or aesthetic reasons. Surface description also plays an important role in the representation of data obtained from medical, geological, physical and other natural phenomena.

In design and engineering the traditional way of representing a surface is to use multiple orthogonal projections. In effect, the surface is defined by a net or mesh of orthogonal plane curves lying in plane sections plus multiple orthogonal projections of certain three-dimensional 'feature' lines (see Fig. 5–1). The curves may originally be designed on paper or they may be taken (digitized) from a three-dimensional model, e.g., the clay stylist's model traditionally used in the automotive industry.

In computer graphics and computer aided design it is advantageous to develop a 'true' three-dimensional mathematical model of a surface. Such a model allows early and relatively easy analysis of surface characteristics, e.g., curvature, or of physical quantities that depend on the surface, e.g., volume, surface area, moment of inertia, etc. Visual rendering of the surface (see Ref. 6–1) for design or design verification is simplified. Further, generation of the necessary information required to fabricate the surface, e.g., numerical control codes, is also considerably simplified as compared to the traditional net of lines approach. Early work by Bézier (Ref. 6–2), Sabin (Ref. 6–3) and Peters (Ref. 6–4) among others demonstrated the feasibility of this approach. Recently surface description techniques have advanced to the point where it is 'almost' possible to abolish the traditional net of lines surface description.

There are two basic philosophies embedded in surface description techniques.

The first, mostly associated with the name of Coons, seeks to create a mathematical surface from known data. The second, mostly associated with the name of Bézier, seeks to create a mathematical surface *ab initio*. Initially, disciplines that depended upon numerical parameters, e.g., engineering, were attracted to the first approach, while disciplines that depended upon visual, tactile or aesthetic factors, e.g., stylists and graphic artists, were attracted to the *ab initio* techniques. Recent work by Rogers (Refs. 6–5 to 6–7) with real-time interactive systems for design of ship hulls and by Cohen (Ref. 6–8) for general surface design shows that the two approaches are compatible.

The elements of mathematical parametric surface representation as used in computer graphics and computer aided design are given in the following sections.

6–2 SURFACES OF REVOLUTION

Perhaps the simplest method for generating a three-dimensional surface is to revolve a two-dimensional entity, e.g., a line or a plane curve, about an axis in space. Such surfaces are called surfaces of revolution. For simplicity, initially the axis of rotation is assumed coincident with the x-axis and in the positive direction. The point, line or plane curve to be rotated is assumed to lie in the xy plane. Later a procedure to remove these restrictions is developed.

The simplest entity that can be rotated about an axis is a point. Provided that the point does not lie on the axis, rotation through an angle of 2π (360°) yields a circle. Rotation through an angle less than 2π (360°) yields a circular arc.

Next in complexity is a line segment parallel to and not coincident with the axis of rotation. Rotation through an angle of 2π (360°) yields a circular cylinder. The radius of the cylinder is the perpendicular distance from the line to the rotation axis. The length of the cylinder is the length of the line segment. An example is shown in Fig. 6–1.

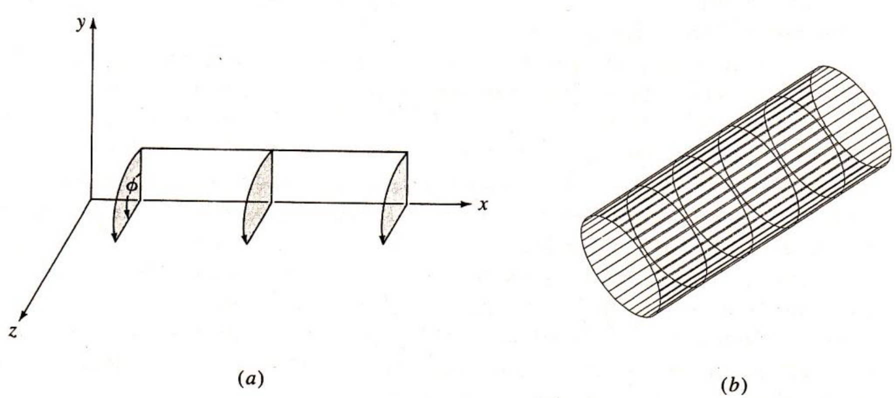

Figure 6–1 Cylindrical surface of revolution. (a) Schematic; (b) result.

If the line segment and the axis of rotation are coplanar and the line segment is not parallel to the rotation axis, then rotation about the axis through 2π (360°) yields a truncated right circular cone. The radius of the cone at each end is the perpendicular distance from the end points of the line segment to the axis of rotation. The length of the cone is the projected length of the line segment on the rotation axis. An example is shown in Fig. 6–2.

Again if the line segment and the axis of rotation are coplanar and the line segment is perpendicular to the axis of rotation, then rotation through 2π (360°) yields a planar disc. If the line segment intersects (or touches) the axis of rotation, a solid disc results; otherwise the disc has a circular hole in it. Examples are shown in Fig. 6–3.

Finally if the line segment is skew to the axis of rotation, i.e., not coplanar, then rotation through 2π (360°) yields a hyperboloid of one sheet (see Secs. 6–4 and 6–7).

Closed or open polygons can also be used to generate surfaces of revolution. An example representing a cone with a cylindrical hole in it is shown in Fig. 6–4.

The parametric equation for a point on a surface of revolution is developed by recalling that the parametric equation of the entity to be rotated, e.g.,

$$P(t) = [\, x(t) \quad y(t) \quad z(t)\,] \qquad 0 \le t \le t_{\max}$$

is a function of the single parameter t. Rotation about an axis causes the location of the point to also be a function of the rotation angle ϕ. Thus, a point on a surface of revolution is specified by *two* parameters t and ϕ. It is a biparametric function as shown in Fig. 6–5.

For the specific case at hand, i.e., rotation about the x-axis of an entity

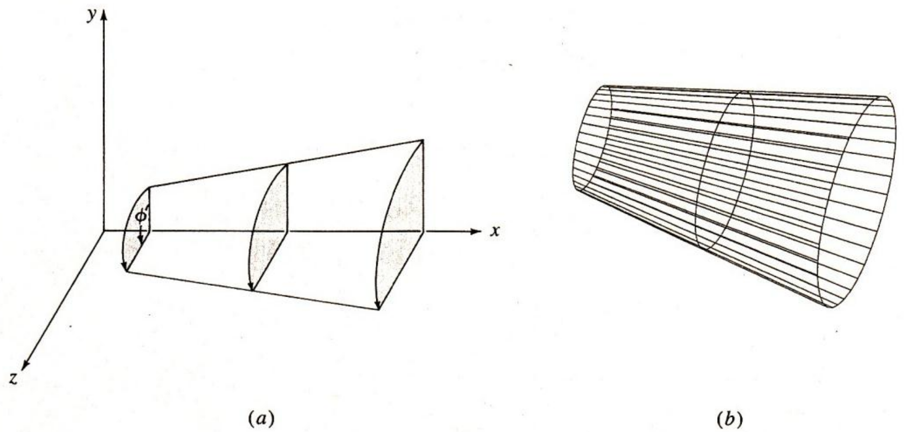

Figure 6–2 Conical surface of revolution. (a) Schematic; (b) result.

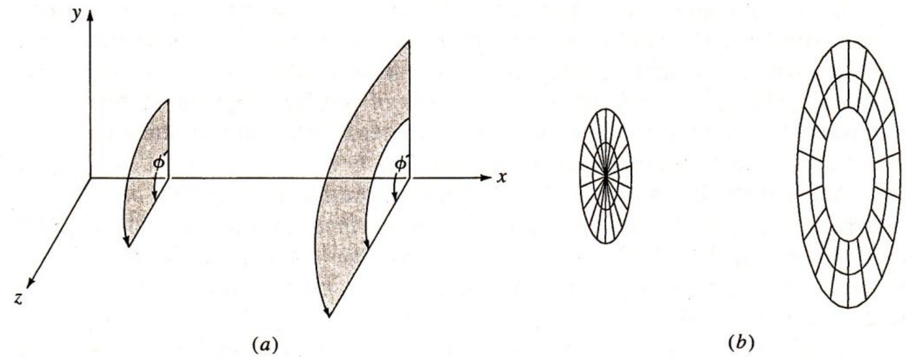

Figure 6–3 A disc as a surface of revolution. (a) Schematic; (b) result.

initially lying in the xy plane, the surface equation is†

$$Q(t,\phi) = [\, x(t) \quad y(t)\cos\phi \quad y(t)\sin\phi \,] \tag{6-1}$$

Note that here the x coordinate does not change. An example is illustrative.

Example 6–1 Simple Surface of Revolution

Consider the line segment with end points $P_1[\,1 \quad 1 \quad 0\,]$ and $P_2[\,6 \quad 2 \quad 0\,]$ lying in the xy plane. Rotating the line about the x-axis yields a conical surface. Determine the point on this surface at $t = 0.5$, $\phi = \pi/3$ (60°).

The parametric equation for the line segment from P_1 to P_2 is

$$P(t) = [\, x(t) \quad y(t) \quad z(t) \,] = P_1 + (P_2 - P_1)t \qquad 0 \le t \le 1$$

with Cartesian components

$$x(t) = x_1 + (x_2 - x_1)t = 1 + 5t$$

$$y(t) = y_1 + (y_2 - y_1)t = 1 + t$$

$$z(t) = z_1 + (z_2 - z_1)t = 0$$

Using Eq. (6–1), the point $Q\,(1/2, \pi/3)$ on the surface of revolution is

$$\begin{aligned} Q(1/2,\pi/3) &= [\, 1+5t \quad (1+t)\cos\phi \quad (1+t)\sin\phi \,] \\ &= \left[\, \frac{7}{2} \quad \frac{3}{2}\cos\left(\frac{\pi}{3}\right) \quad \frac{3}{2}\sin\left(\frac{\pi}{3}\right) \right] \\ &= \left[\, \frac{7}{2} \quad \frac{3}{4} \quad \frac{3\sqrt{3}}{4} \,\right] = [\, 3.5 \quad 0.75 \quad 1.3 \,] \end{aligned}$$

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†Note that $Q(t,\phi)$ is a vector valued function. Thus, in vector form

$$Q(t,\phi) = x(t)\mathbf{i} + y(t)\cos\phi\,\mathbf{j} + y(t)\sin\phi\,\mathbf{k}$$

where **i**, **j**, **k** are unit vectors in the x, y, z directions, respectively.

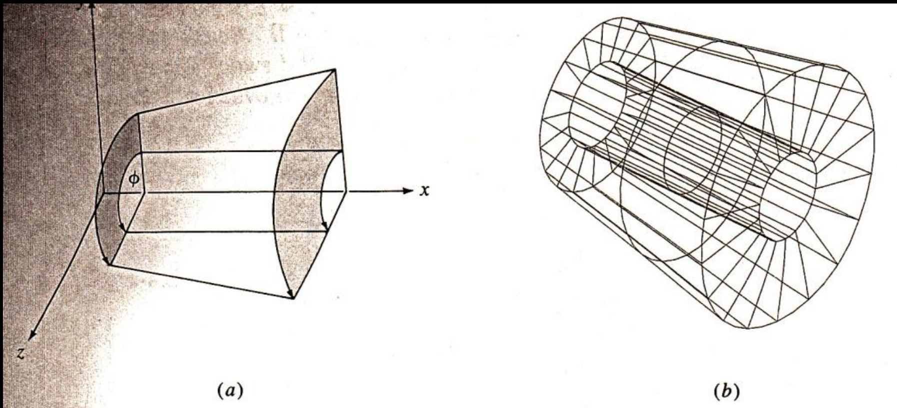


Figure 6–4 A surface of revolution from a closed polygon. (a) Schematic; (b) result.

Rotating plane curves also yields surfaces of revolution. A sphere is obtained by rotating an origin-centered semicircle in the xy plane about the x-axis as shown in Fig. 6–6a. Recalling the parametric equation of the circle (see Sec. 4–5)

$$x = r\cos\theta \qquad 0 \le \theta \le \pi \qquad (4-4)$$
$$y = r\sin\theta$$

the parametric equation of the sphere is

$$Q(\theta,\phi) = [\, x(\theta) \quad y(\theta)\cos\phi \quad y(\theta)\sin\phi \,]$$
$$= [\, r\cos\theta \quad r\sin\theta\cos\phi \quad r\sin\theta\sin\phi \,] \qquad 0 \le \theta \le \pi$$
$$0 \le \phi \le 2\pi \qquad (6-2)$$

An ellipsoid of revolution is obtained if the parametric equation of an origin centered semiellipse in the xy plane is substituted for that of the circle. Specifically recalling the parametric equation of the semiellipse (see Sec. 4–6)

$$x = a\cos\theta \qquad 0 \le \theta \le \pi \qquad (4-6)$$
$$y = b\sin\theta$$

gives the parametric equation for any point on the ellipsoid of revolution as

$$Q(\theta,\phi) = [\, a\cos\theta \quad b\sin\theta\cos\phi \quad b\sin\theta\sin\phi \,] \qquad 0 \le \theta \le \pi$$
$$0 \le \phi \le 2\pi \qquad (6-3)$$

If $a = b = r$, then Eq. (6–3) reduces to Eq. (6–2) for a sphere. An ellipsoid of revolution is shown in Fig. 6–6b.

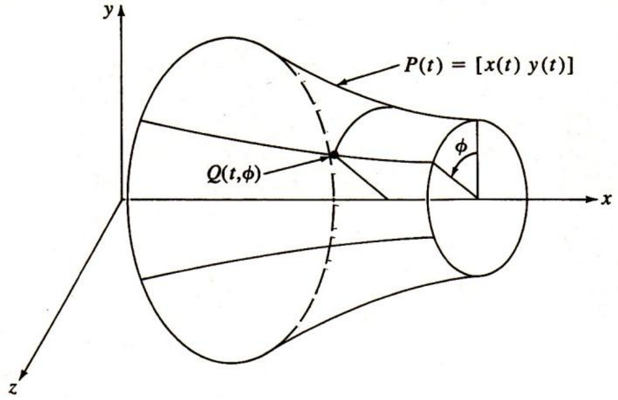


Figure 6–5 Biparametric surface of revolution.

If the axis of rotation does not pass through the center of a complete circle or ellipse, then rotation about the axis generates a torus with a circular or elliptical cross section as appropriate. Noting that the parametric equation of a non-origin-centered ellipse in the xy plane is

$$x = h + a\cos\theta \qquad 0 \le \theta \le 2\pi$$

$$y = k + b\sin\theta$$

where (h, k) are the x and y coordinates of the center of the ellipse, the parametric equation for any point on the torus is

$$Q(\theta, \phi) = [\, h + a\cos\theta \quad (k + b\sin\theta)\cos\phi \quad (k + b\sin\theta)\sin\phi \,] \qquad (6-4)$$

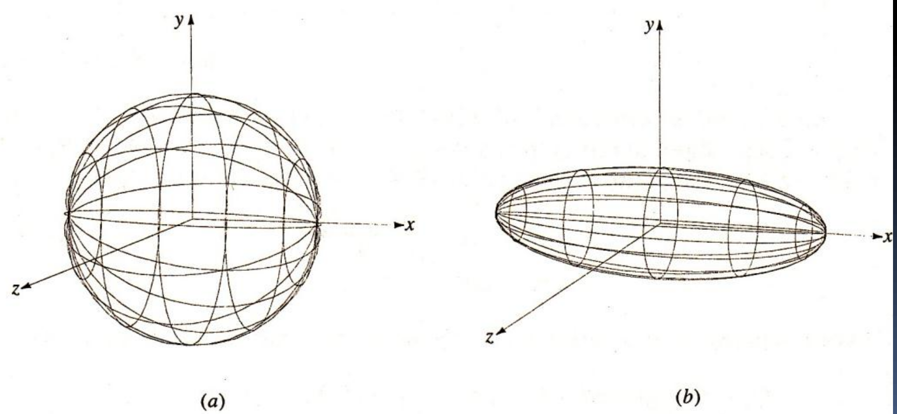


Figure 6–6 Surfaces of revolution. (a) Sphere; (b) ellipsoid.

where $0 \le \theta \le 2\pi$, $0 \le \phi \le 2\pi$. If $a = b = r$, Eq. (6–4) yields a torus with a circular cross section. If $a \ne b$, then a torus with an elliptical cross section results. Figure 6–7 shows both a circular and an elliptical cross section torus.

A paraboloid of revolution is obtained by rotating the parametric parabola (see Sec. 4–7)

$$x = a\theta^2 \qquad 0 \le \theta \le \theta_{\max} \qquad (4-9)$$
$$y = 2a\theta$$

about the x-axis. The parametric surface is given by

$$Q(\theta, \phi) = [\, a\theta^2 \quad 2a\theta\cos\phi \quad 2a\theta\sin\phi \,] \qquad \begin{array}{l} 0 \le \theta \le \theta_{\max} \\ 0 \le \phi \le 2\pi \end{array} \qquad (6-5)$$

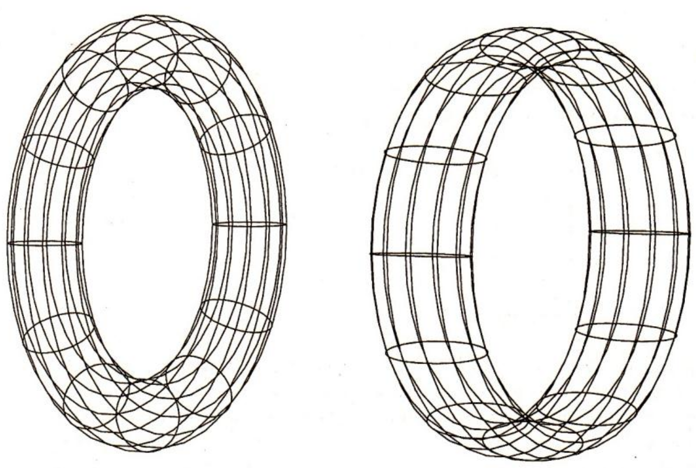
A hyperboloid of revolution is obtained by rotating the parametric hyperbola

$$x = a\sec\theta \qquad 0 \le \theta \le \theta_{\max} \qquad (4-14)$$
$$y = b\tan\theta$$

about the x-axis. The parametric surface is given by

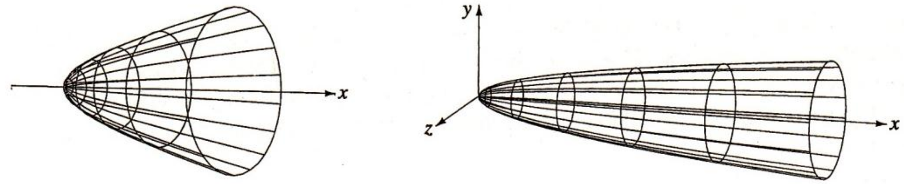
$$Q(\theta, \phi) = [\, a\sec\theta \quad b\tan\theta\cos\phi \quad b\tan\theta\sin\phi \,] \qquad \begin{array}{l} 0 \le \theta \le \theta_{\max} \\ 0 \le \phi \le 2\pi \end{array} \qquad (6-6)$$

Examples are shown in Fig. 6–8.



(a) (b)

Figure 6–7 Tori. (a) Circular cross section; (b) elliptical cross section.



(a) (b)

Figure 6–8 Surfaces of revolution. (a) Paraboloid; (b) hyperboloid.

Any parametric curve can be used to create a surface of revolution. Obvious possibilities are cubic spline, parabolically blended, Bézier and B-spline curves. Figure 6–9 shows a surface of revolution created using a relatively simple parabolically blended curve. Figure 6–10 shows a handleless mug created as a surface of revolution using a complex open B-spline curve. Notice that the mug has both an inside and an outside. Here rotation is about the y-axis.†

Recall that in matrix form a parametric space curve (see Eqs. (5–27), (5–44), (5–67) and (5–94)) is given by

$$P(t) = [\,T\,][\,N\,][\,G\,]$$

where $[\,T\,]$, $[\,N\,]$ and $[\,G\,]$ are parameter, blending function and geometry matrices, respectively. The general form of the matrix equation for a surface of revolution is thus

$$Q(t,\phi) = [\,T\,][\,N\,][\,G\,][\,S\,] \qquad (6-7)$$

where $[\,S\,]$ represents the contribution due to rotation about an axis by the angle ϕ. For the specific case of rotation about the x-axis

$$[\,S\,] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (6-8)$$

An example illustrates these techniques.

† The method used to design the B-spline curve for the mug shown in Fig. 6–10 may be of interest. A sketch of a pottery mug used by the first author was first made on graph paper. Thirty-four points were digitized from the sketch. An initial B-spline polygon ($k=4, n+1=21$) was derived using the fitting technique described in Sec. 5–11. The defining polygon vertices were then transferred to a real time interactive graphics system for final design (see Ref. 6–7). A B-spline curve created from the final 25 polygon vertices was used to generate the surface of revolution shown. The defining polygon vertices are [0 0.684], [0.302 0.684], [0.302 0.684], [1.598 −0.288], [1.088 0.405], [0.374 0.773], [0.848 0.993], [1.232 1.446], [1.451 1.875], [1.502 2.631], [1.448 3.308], [1.226 4.076], [1.397 4.449], [1.398 4.592], [1.268 4.572], [1.241 4.485], [1.109 4.170], [1.214 3.669], [1.313 2.889], [1.310 2.280], [1.172 1.575], [0.629 1.119], [0.281 0.882], [0.278 0.891], [0 0.897].

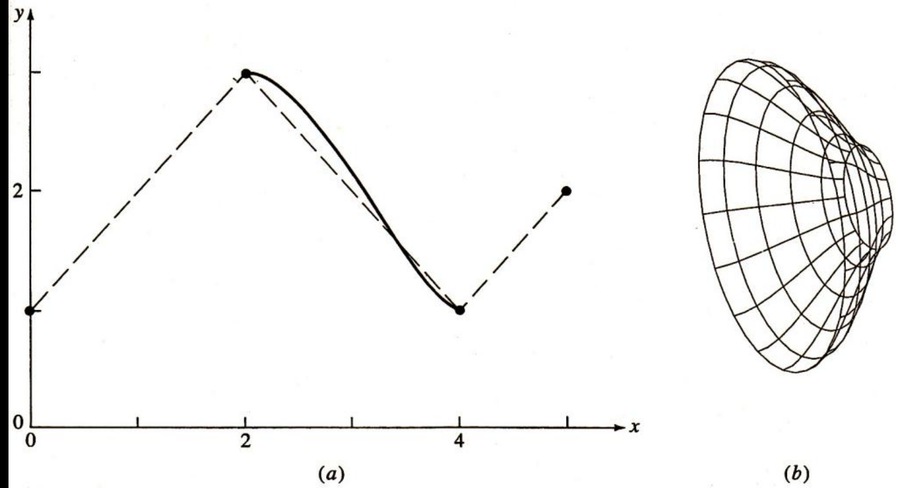


Figure 6–9 Parabolically blended surface of revolution. (a) Generating curve; (b) surface.

Example 6–2 Parabolically Blended Surface of Revolution

Consider the parabolically blended curve defined by the points $P_1[\,0 \quad 1 \quad 0\,]$, $P_2[\,2 \quad 3 \quad 0\,]$, $P_3[\,4 \quad 1 \quad 0\,]$, $P_4[\,5 \quad 2 \quad 0\,]$. Rotate this curve about the x-axis through 2π to obtain a surface of revolution. Calculate the surface point at $t = 0.5$, $\phi = \pi/3$ (60°).

Using Eqs. (6–7) and (6–8) the parametric equation of the surface of revolution is

$$Q(t,\phi) = [\,T\,][\,A\,][\,G\,][\,S\,]$$

where $[\,S\,]$, $[\,T\,]$, $[\,A\,]$ and $[\,G\,]$ are given by Eqs. (5–44),(5–52) and (5–53), respectively.

Specifically

$$Q(t,\phi) = \left(\frac{1}{2}\right)[\,t^3 \quad t^2 \quad t \quad 1\,]\begin{bmatrix} -1 & 3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 2 & 3 & 0 & 1 \\ 4 & 1 & 0 & 1 \\ 5 & 2 & 0 & 1 \end{bmatrix} \times$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Q(t,\phi) = \left(\frac{1}{2}\right)[\,t^3 \quad t^2 \quad t \quad 1\,]\begin{bmatrix} -1 & 7 & 0 & 1 \\ 1 & -11 & 0 & 1 \\ 4 & 0 & 0 & 1 \\ 4 & 6 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

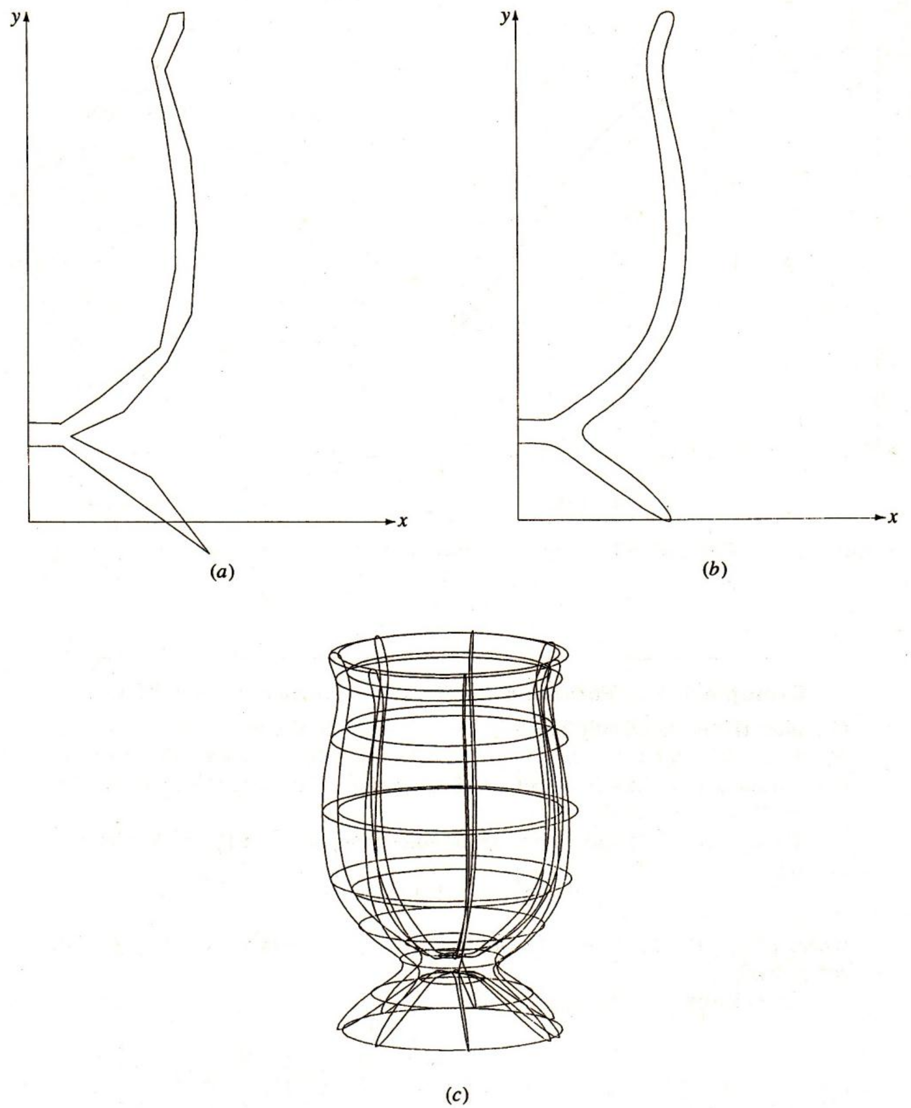


Figure 6–10 B-spline surface of revolution. (a) Polygon vertices; (b) B-spline curve; (c) surface.

For $t = 0.5$ and $\phi = \pi/3$ (60°)

$$Q(0.5, \pi/3) = \frac{1}{2}\begin{bmatrix} \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} -1 & 7 & 0 & 1 \\ 1 & -11 & 0 & 1 \\ 4 & 0 & 0 & 1 \\ 4 & 6 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotate about the y-axis by $+90°$ to make the a-axis coincident with the x-axis.†

The above three steps are needed only to determine the inverse transformations required to place the surface of revolution correctly in three space. Having generated the surface of revolution by rotation about the x-axis, the following three steps correctly place it in three space:

Translate in x to place the center of the surface of revolution at the correct location on the a-axis.

Apply the inverse of the combined rotation transformations to the surface of revolution.

Perform the inverse of the translation of the point a_0 to the surface of revolution.

A point on the surface of revolution is then given by

$$Q(t, \phi) = [\,\bar{Q}\,][\,Tr\,][\,\bar{R}_y\,]^{-1}[\,R_y\,]^{-1}[\,R_x\,]^{-1}[\,Tr\,]^{-1} \qquad (6-9)$$

where $[\,Tr\,]$, $[\,R_x\,]$, $[\,R_y\,]$ are given by Eqs. (3–22) to (3–24). $[\,\bar{R}_y\,]^{-1}$ is given by Eq. (3–8), and $[\,\bar{Q}\,]$ is given in the form of Eq. (6–7) with the geometry in $[\,G\,]$ represented in homogeneous coordinates. $[\,S\,]$ is now a 4×4 matrix given by

$$[\,S\,] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (6-10)$$

An example illustrates the procedure.

Example 6–3 Surface of Revolution About an Arbitrary Axis

Determine the point at $t = \pi/2$, $\phi = \pi/6$ on a surface of revolution formed by rotating an ellipse with major axis inclined to the axis of revolution. The axis of revolution passes through the center of the ellipse and lies in the plane of the ellipse. The angle of inclination is $i = \pi/4$. The semimajor and semiminor axes are $a = 5$, $b = 1$, respectively. The axis passes through the points $a_0\,[\,0 \quad 10 \quad 10\,]$ and $a_1\,[\,10 \quad 10 \quad 0\,]$. The center of the ellipse is at a_1.

First the direction cosines of the axis of rotation are (see Eq. 3–26)

$$[\,c_x \quad c_y \quad c_z\,] = [\,1/\sqrt{2} \quad 0 \quad -1/\sqrt{2}\,]$$

and (see Eq. 3–18)

$$d = \sqrt{{c_y}^2 + {c_z}^2} = 1/\sqrt{2}$$

Thus, using Eqs. (3–22) to (3–24)

†This step is necessary *only* to be consistent with previous work. The surface of revolution could as easily be generated about the z-axis.

Rotate about the y-axis by $+90°$ to make the a-axis coincident with the x-axis.†

The above three steps are needed only to determine the inverse transformations required to place the surface of revolution correctly in three space. Having generated the surface of revolution by rotation about the x-axis, the following three steps correctly place it in three space:

Translate in x to place the center of the surface of revolution at the correct location on the a-axis.

Apply the inverse of the combined rotation transformations to the surface of revolution.

Perform the inverse of the translation of the point a_0 to the surface of revolution.

A point on the surface of revolution is then given by

$$Q(t,\phi) = [\,\bar{Q}\,][\,Tr\,][\,\bar{R}_y\,]^{-1}[\,R_y\,]^{-1}[\,R_x\,]^{-1}[\,Tr\,]^{-1} \qquad (6-9)$$

where $[\,Tr\,]$, $[\,R_x\,]$, $[\,R_y\,]$ are given by Eqs. (3–22) to (3–24). $[\,\bar{R}_y\,]^{-1}$ is given by Eq. (3–8), and $[\,\bar{Q}\,]$ is given in the form of Eq. (6–7) with the geometry in $[\,G\,]$ represented in homogeneous coordinates. $[\,S\,]$ is now a 4×4 matrix given by

$$[\,S\,] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (6-10)$$

An example illustrates the procedure.

Example 6–3 Surface of Revolution About an Arbitrary Axis

Determine the point at $t=\pi/2$, $\phi=\pi/6$ on a surface of revolution formed by rotating an ellipse with major axis inclined to the axis of revolution. The axis of revolution passes through the center of the ellipse and lies in the plane of the ellipse. The angle of inclination is $i=\pi/4$. The semimajor and semiminor axes are $a=5$, $b=1$, respectively. The axis passes through the points $a_0\,[\,0 \quad 10 \quad 10\,]$ and $a_1\,[\,10 \quad 10 \quad 0\,]$. The center of the ellipse is at a_1.

First the direction cosines of the axis of rotation are (see Eq. 3–26)

$$[\,c_x \quad c_y \quad c_z\,] = [\,1/\sqrt{2} \quad 0 \quad -1/\sqrt{2}\,]$$

and (see Eq. 3–18)

$$d = \sqrt{{c_y}^2 + {c_z}^2} = 1/\sqrt{2}$$

Thus, using Eqs. (3–22) to (3–24)

†This step is necessary *only* to be consistent with previous work. The surface of revolution could as easily be generated about the z-axis.

$$[M_1] = [Tr][R_x][R_y]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -10 & -10 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 1 & 0 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & -1 & 0 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ -10/\sqrt{2} & 10 & 10/\sqrt{2} & 1 \end{bmatrix}$$

makes the rotation axis coincident with the z-axis. Rotating about y by 90° yields

$$[M_2] = [Tr][R_x][R_y][\bar{R}_y]$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & -1 & 0 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ -10/\sqrt{2} & 10 & 10/\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & -1 & 0 & 0 \\ -1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 10/\sqrt{2} & 10 & 10/\sqrt{2} & 1 \end{bmatrix}$$

Using $[M_2]$ and the homogeneous coordinates, the center of the ellipse originally at a is now

$$[10 \quad 10 \quad 10 \quad 1] \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & -1 & 0 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ -10/\sqrt{2} & 10 & 10/\sqrt{2} & 1 \end{bmatrix} = [20/\sqrt{2} \quad 0 \quad 0 \quad 1]$$

i.e., at $h = 20/\sqrt{2}$ on the x-axis.

Recalling Ex. 4–4 of Sec. 4–6, rotating an origin-centered ellipse about the origin by the angle i yields the parametric equations

$$x = a\cos t\cos i - b\sin t\sin i \qquad 0 \le t \le 2\pi$$

$$y = a\cos t\sin i + b\sin t\cos i$$

which may be written in the form of Eq. (6–7) as

$$[\cos t \quad \sin t \quad 0 \quad 1] \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos i & \sin i & 0 & 0 \\ -\sin i & \cos i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The surface of revolution is then

$$[Q] = [T][N][G][S]$$

$$= [\cos t \quad \sin t \quad 0 \quad 1]\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos i & \sin i & 0 & 0 \\ -\sin i & \cos i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For $a = 5$, $b = 1$, $i = \pi/4$

$$[\bar{Q}] = [\cos t \quad \sin t \quad 0 \quad 1]\begin{bmatrix} 5\sqrt{2}/2 & 5\sqrt{2}/2 & 0 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This origin-centered surface of revolution is also shown in Fig. 6–10.

Translating the origin to (h, n) on the x-axis and noting that

$$[M_2]^{-1} = [\bar{R}_y]^{-1}[R_y]^{-1}[R_x]^{-1}[Tr]^{-1}$$

$$= \begin{bmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 & 0 \\ 0 & -1 & 0 & 0 \\ -\sqrt{2}/2 & 0 & -\sqrt{2}/2 & 0 \\ 0 & 10 & 10 & 1 \end{bmatrix}$$

yields $$[Q] = [T][N][G][S][Tr_x][M_2]^{-1}$$

The point at $Q(t, \phi)$ is

$$Q(t,\phi) = [\cos t \quad \sin t \quad 0 \quad 1]\begin{bmatrix} 5\sqrt{2}/2 & 5\sqrt{2}/2 & 0 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 20/\sqrt{2} & 0 & 0 & 1 \end{bmatrix} \times$$

$$\begin{bmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 & 0 \\ 0 & -1 & 0 & 0 \\ -\sqrt{2}/2 & 0 & -\sqrt{2}/2 & 0 \\ 0 & 10 & 10 & 1 \end{bmatrix}$$

or

$$Q(t,\phi) = \left(\frac{1}{2}\right) [\,\cos t \quad \sin t \quad 0 \quad 1\,] \times$$

$$\begin{bmatrix} 5(1-\sin\phi) & -5\sqrt{2}\cos\phi & -5(1+\sin\phi) & 0 \\ -(1+\sin\phi) & -\sqrt{2}\cos\phi & (1-\sin\phi) & 0 \\ 0 & 0 & 0 & 0 \\ 20 & 20 & 0 & 0 \end{bmatrix}$$

For $t = \pi/2$, $\phi = \pi/6$

$$Q(\pi/2,\ \pi/6) = \left(\frac{1}{2}\right) [\,0 \quad 1 \quad 0 \quad 1\,] \begin{bmatrix} 5/2 & -5\sqrt{6}/2 & -15/2 & 0 \\ -3/2 & -\sqrt{6}/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 20 & 20 & 0 & 1 \end{bmatrix}$$

$$= [\,37/4 \quad 10-\sqrt{6}/4 \quad 1/4 \quad 1\,]$$

$$= [\,9.25 \quad 9.388 \quad 0.25 \quad 1\,]$$

The resulting surface is shown in Fig. 6–12. Notice that the surface is both self-intersecting and complex.

Formally differentiating Eq. (6–7) yields the parametric derivatives for a surface of revolution. Specifically, the derivative in the axial direction is

$$Q_t(t,\phi) = [\,T'\,][\,N\,][\,G\,][\,S\,] \tag{6-11}$$

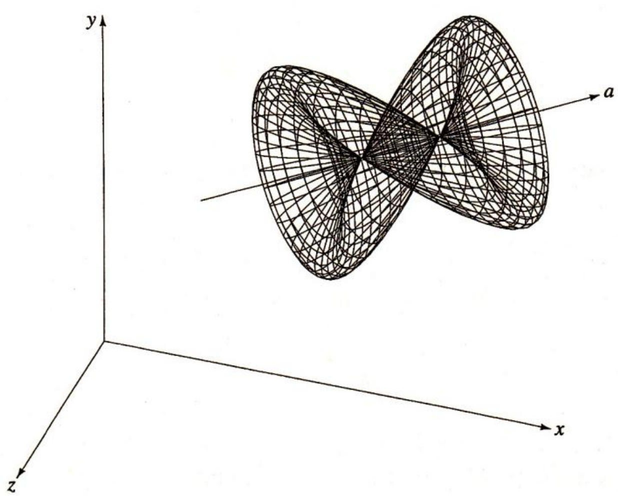


Figure 6–12 Complex elliptical surface of revolution for Ex. 6–3.

and in the radial direction

$$Q_\phi(t,\phi) = [\,T\,][\,N\,][\,G\,][\,S'\,] \tag{6-12}$$

where the prime denotes appropriate differentiation.

The surface normal is given by the cross product of the parametric derivatives, i.e.,

$$n = Q_t \times Q_\phi \tag{6-13}$$

6–3 SWEEP SURFACES

A three-dimensional surface is also obtained by traversing an entity, e.g., a line, polygon or curve, along a path in space. The resulting surfaces are called sweep surfaces. Sweep surface generation is frequently used in geometric modeling. The simplest sweep entity is a point. The result of sweeping a point along a path is, of course, not a surface but a space curve. However, it serves to illustrate the fundamental technique.

Consider the position vector $P[\,x \quad y \quad z \quad 1\,]$ swept along the path represented by the sweep transformation $[\,T(s)\,]$. The position vector $Q(s)$ representing the resulting curve is given by

$$Q(s) = P[\,T(s)\,] \qquad s_1 \le s \le s_2 \tag{6-14}$$

The transformation $[\,T(s)\,]$ determines the shape of the curve. For example, if the path is a straight line of length n parallel to the z-axis, then (see Eq. 3–14)

$$[\,T(s)\,] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & ns & 1 \end{bmatrix} \qquad 0 \le s \le 1$$

If the path is an origin-centered circle in a $z =$ constant plane, then (see Eq. 3–8)

$$[\,T(s)\,] = \begin{bmatrix} \left(\frac{r}{x}\right)\cos\{2\pi(s+s_i)\} & 0 & 0 & 0 \\ 0 & \left(\frac{r}{y}\right)\sin\{2\pi(s+s_i)\} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad 0 \le s \le 1$$

where $s_i = (1/2\pi)\tan^{-1}(y_i/x_i)$ and for $P[\,x \quad y \quad z \quad 1\,]$, $r = \sqrt{x^2+y^2}$. Here, the subscript i is used to indicate the initial or starting point.

Complex paths can be developed by combining simple paths. For example, combining the two previous path transformations yields a single turn of a helical

Here the sweep transformation matrix is a translation followed by a rotation given by†

$$[\,T(s)\,] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(2\pi s) & \sin(2\pi s) & 0 \\ 0 & -\sin(2\pi s) & \cos(2\pi s) & 0 \\ ls & 0 & 0 & 1 \end{bmatrix}$$

The parametric equation of the line segment is

$$P(t) = P_1 + (P_2 - P_1)t = [\,0 \quad 0 \quad 0 \quad 1\,] + [\,0-0 \quad 3-0 \quad 0-0 \quad 1-1\,]\,t$$
$$= [\,0 \quad 3t \quad 0 \quad 1\,]$$

From Eq. (6–15) the sweep surface is given by

$$Q(t,s) = [\,P(t)\,][\,T(s)\,]$$
$$= [\,0 \quad 3t \quad 0 \quad 1\,] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(2\pi s) & \sin(2\pi s) & 0 \\ 0 & -\sin(2\pi s) & \cos(2\pi s) & 0 \\ ls & 0 & 0 & 1 \end{bmatrix}$$

and

$$Q(0.5,\,0.5) = [\,0 \quad 1.5 \quad 0 \quad 1\,] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 5 & 0 & 0 & 1 \end{bmatrix}$$
$$= [\,5 \quad -1.5 \quad 0 \quad 1\,]$$

Complete results are shown in Fig. 6–13.

Parametric curves, e.g., cubic splines, parabolically blended, Bézier and B-spline curves, are also used to generate sweep surfaces. The surface equation is identical to Eq. (6–16) where now $P(t)$ represents the parametric curve. Figure 6–14 shows a sweep surface generated from a single cubic spline curve segment swept parallel to the z-axis. An example illustrates the technique.

Example 6–5 Cubic Spline Sweep Surface

Sweep the normalized cubic spline curve segment defined by $P_1\,[\,0 \quad 3 \quad 0 \quad 1\,]$, $P_2\,[\,3 \quad 0 \quad 0 \quad 1\,]$, $P_1'\,[\,3 \quad 0 \quad 0 \quad 0\,]$, $P_2'\,[\,3 \quad 0 \quad 0 \quad 0\,]$‡ 10 units along the z-axis.

The sweep surface is given by

$$Q(t,s) = [\,C(t)\,][\,T(s)\,] \qquad 0 \le t \le 1, \quad 0 \le s \le 1$$

---

†The same matrix results from a rotation followed by a translation.

‡In homogeneous coordinates a tangent vector has a zero homogeneous coordinate factor.

Here the sweep transformation matrix is a translation followed by a rotation given by[†]

$$[\,T(s)\,] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(2\pi s) & \sin(2\pi s) & 0 \\ 0 & -\sin(2\pi s) & \cos(2\pi s) & 0 \\ ls & 0 & 0 & 1 \end{bmatrix}$$

The parametric equation of the line segment is

$$\begin{aligned} P(t) = P_1 + (P_2 - P_1)t &= [\,0 \quad 0 \quad 0 \quad 1\,] + [\,0-0 \quad 3-0 \quad 0-0 \quad 1-1\,]t \\ &= [\,0 \quad 3t \quad 0 \quad 1\,] \end{aligned}$$

From Eq. (6–15) the sweep surface is given by

$$\begin{aligned} Q(t,s) &= [\,P(t)\,][\,T(s)\,] \\ &= [\,0 \quad 3t \quad 0 \quad 1\,] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(2\pi s) & \sin(2\pi s) & 0 \\ 0 & -\sin(2\pi s) & \cos(2\pi s) & 0 \\ ls & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} Q(0.5,\ 0.5) &= [\,0 \quad 1.5 \quad 0 \quad 1\,] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 5 & 0 & 0 & 1 \end{bmatrix} \\ &= [\,5 \quad -1.5 \quad 0 \quad 1\,] \end{aligned}$$

Complete results are shown in Fig. 6–13.

Parametric curves, e.g., cubic splines, parabolically blended, Bézier and B-spline curves, are also used to generate sweep surfaces. The surface equation is identical to Eq. (6–16) where now $P(t)$ represents the parametric curve. Figure 6–14 shows a sweep surface generated from a single cubic spline curve segment swept parallel to the z-axis. An example illustrates the technique.

Example 6–5 Cubic Spline Sweep Surface

Sweep the normalized cubic spline curve segment defined by $P_1\,[\,0 \quad 3 \quad 0 \quad 1\,]$, $P_2\,[\,3 \quad 0 \quad 0 \quad 1\,]$, $P_1'\,[\,3 \quad 0 \quad 0 \quad 0\,]$, $P_2'\,[\,3 \quad 0 \quad 0 \quad 0\,]$[‡] 10 units along the z-axis.

The sweep surface is given by

$$Q(t,s) = [\,C(t)\,][\,T(s)\,] \qquad 0 \le t \le 1, \quad 0 \le s \le 1$$

[†] The same matrix results from a rotation followed by a translation.

[‡] In homogeneous coordinates a tangent vector has a zero homogeneous coordinate factor.

The normalized cubic spline segment is given by (see Eq. 5–27)

$$[C(t)] = [t^3 \quad t^2 \quad t \quad 1]\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 & 0 & 1 \\ 3 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}$$

The curve segment is shown in Fig. 6–14a.

The sweep transformation is (see Eq. 3–14)

$$[T(s)] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & ns & 1 \end{bmatrix}$$

Hence,

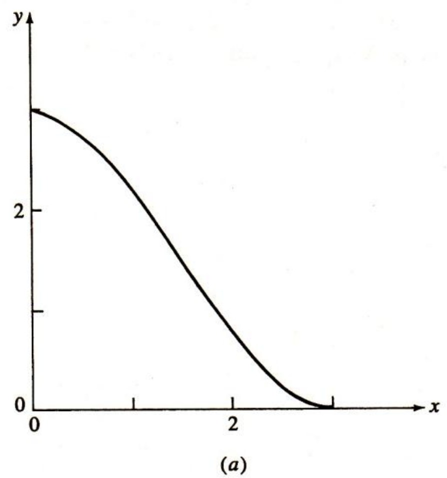
$$Q(t,s) = [t^3 \quad t^2 \quad t \quad 1]\begin{bmatrix} 0 & 6 & 0 & 0 \\ 6 & -9 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 10s & 1 \end{bmatrix}$$

For $t = 0.5$, $s = 0.5$

$$Q(0.5,\ 0.5) = [0.125 \quad 0.25 \quad 0.5 \quad 1]\begin{bmatrix} 0 & 6 & 0 & 0 \\ 6 & -9 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 1 \end{bmatrix}$$

$$= [3 \quad 1.5 \quad 5 \quad 1]$$

Complete results are shown in Fig. 6–14b.



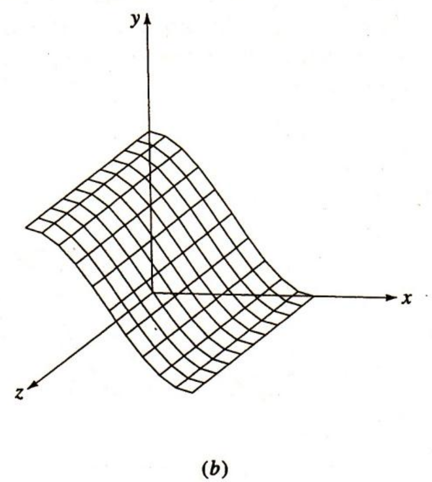


Figure 6–14 A cubic spline based sweep surface. (a) Curve; (b) surface.

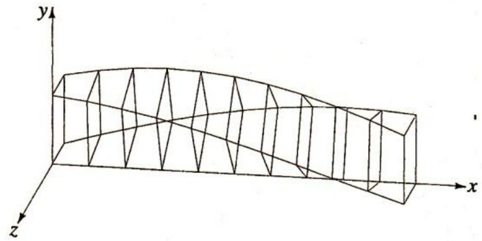


Figure 6–16 Sweep surface generated by a square simultaneously sweeping along and rotating about the x-axis.

Example 6–6 Complex Sweep Surface

Sweep the planar square defined by vertices $P_1\,[\,0 \quad -1 \quad 1\,]$, $P_2\,[\,0 \quad -1 \quad -1\,]$, $P_3\,[\,0 \quad 1 \quad -1\,]$, $P_4\,[\,0 \quad 1 \quad 1\,]$ along the path $x = 10s$, $y = \cos(\pi s) - 1$ while maintaining the normal to the polygon in the instantaneous direction of the tangent to the path.

The instantaneous direction of the path tangent is $[\,10 \quad -\pi\sin(\pi s) \quad 0\,]$. The rotation angle about the z-axis to align the polygon normal with the tangent to the path is thus

$$\psi = \tan^{-1}\left(\frac{-\pi\sin(\pi s)}{10}\right)$$

The sweep transformation is thus

$$[\,T(s)\,] = \begin{bmatrix} \cos\psi & \sin\psi & 0 & 0 \\ -\sin\psi & \cos\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 10s & \cos(\pi s) - 1 & 0 & 1 \end{bmatrix}$$

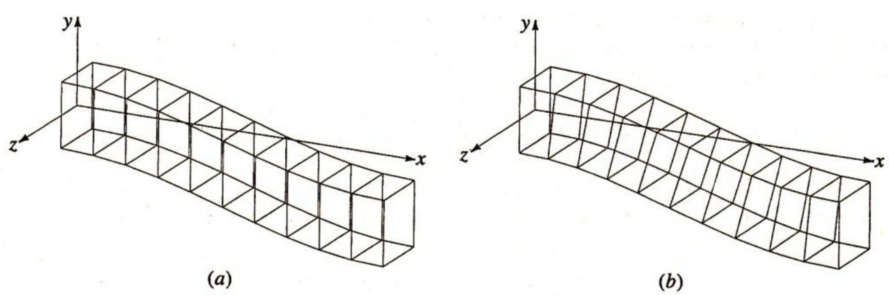


Figure 6–17 A polygon swept along a path. (a) Normal in the x direction; (b) normal in the instantaneous tangent vector direction.

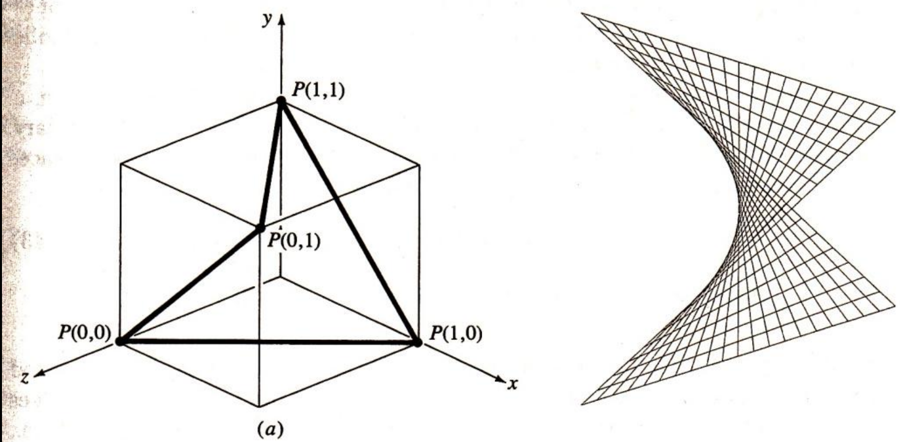


Figure 6–26 Bilinear surface. (a) Defining corner points; (b) surface.

6–8 RULED AND DEVELOPABLE SURFACES

Ruled surfaces are frequently used in both the aircraft and the shipbuilding industries. For example, most aircraft wings are cylindrical ruled surfaces. Technically a ruled surface is generated by a straight line moving along a path with one degree of freedom. Alternately a ruled surface is identified using the following technique. At any point on the surface, rotate a plane containing the normal to the surface at that point about the normal (see Fig. 6–27). If in at least one orientation every point on the edge of the plane contacts the surface, the surface is ruled in that direction. If the edge of the rotating plane completely touches the surface in more than one orientation, the surface is multiply ruled at that point.

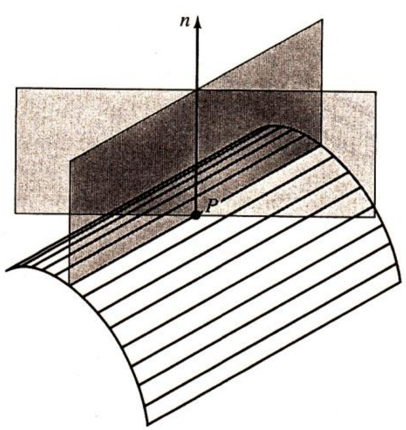


Figure 6–27 Characteristics of a ruled surface.

and $$Q(0.5,0.5) = P(0,0.5)(1-0.5) + P(1,0.5)(0.5)$$
$$= 0.5[\,1.125 \quad 1 \quad 0\,] + 0.5[\,1.5 \quad 1 \quad 6\,]$$
$$= [\,1.3125 \quad 1 \quad 3\,]$$

Complete results are shown in Fig. 6–28. The point on the surface corresponding to $Q(0.5,0.5)$ is marked with a dot. Notice how the curve containing the cusp, $P(0,w)$, is smoothly blended into the continuous curve $P(1,w)$.

Of particular practical interest is whether a ruled surface is developable. Not all ruled surfaces are developable. However, all developable surfaces are ruled surfaces. If a surface is developable, then, by a succession of small rotations of the surface about the generating line, the surface can be unfolded or developed onto a plane without stretching or tearing. Developable surfaces are of considerable importance to sheet-metal- or plate-metal-based industries and to a less extent to fabric-based industries.

It is clear that among the ruled quadric surfaces both the cones and cylinders are developable. However, a few moments' reflection confirms that neither the hyperbola of one sheet (see Fig. 6–18d) nor the hyperbolic paraboloid (see Fig. 6–26), both of which are ruled surfaces, is a developable surface.

To determine if a surface or a portion of a surface is developable, it is necessary to consider the curvature of a parametric surface. At any point P on a surface, the curve of intersection of a plane containing the normal to the surface at P and the surface has a curvature κ (see Fig. 6–29). As the plane is rotated about the normal, the curvature changes. Euler, the great Swiss mathematician, showed that unique directions for which the curvature is a minimum and a maximum exists. The curvatures in these directions are called the principal curvatures, κ_{min} and κ_{max}. Further, the principal curvature directions are orthogonal. Two combinations of the principal curvatures are of particular interest, the average and the Gaussian curvatures:

$$H = \frac{\kappa_{\text{min}} + \kappa_{\text{max}}}{2} \qquad (6-45)$$

$$K = \kappa_{\text{min}}\kappa_{\text{max}} \qquad (6-46)$$

For a developable surface the Gaussian curvature K is everywhere zero, i.e., $K = 0$. Dill (Ref. 6–18) has shown that for biparametric surfaces the average and Gaussian curvatures are given by†

$$H = \frac{A|Q_w|^2 - 2BQ_u \cdot Q_w + C|Q_u|^2}{2|Q_u \times Q_w|^3} \qquad (6-47)$$

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†Here subscript notation is used for partial derivatives, i.e., $Q_u = \partial Q/\partial u$ etc.

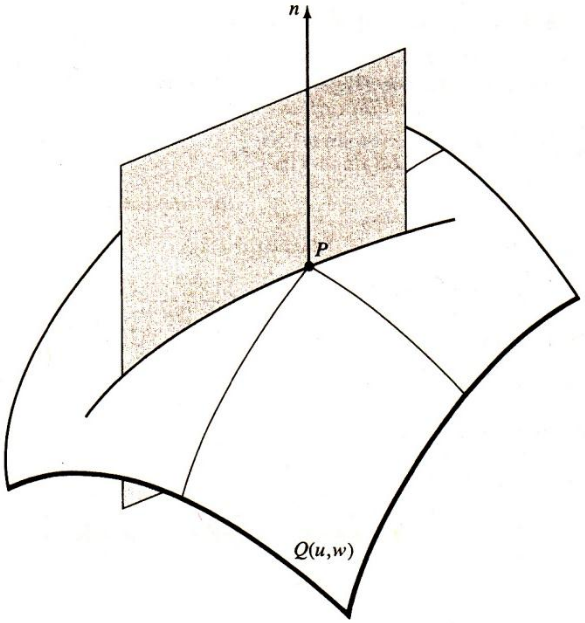

Figure 6–29 Curvature of a biparametric surface.

$$K = \frac{AC - B^2}{|Q_u \times Q_w|^4} \qquad (6-48)$$

where

$$(A\,B\,C) = [\,Q_u \times Q_w\,] \cdot [\,Q_{uu} \quad Q_{uw} \quad Q_{ww}\,]$$

As shown in Table 6–2 the sign of the Gaussian curvature serves to characterize the local shape of the surface: elliptic, hyperbolic, cylindrical or conical. Since the Gaussian curvature of a developable surface must be zero, the surface must be composed of cylindrical, conical or planar pieces. An example helps to illustrate this discussion.

Table 6–2 Surface Types

$\kappa_{\min}\kappa_{\max}$	K	Shape
Same sign	> 0	Elliptic (bump or hollow)
Opposite sign	< 0	Hyperbolic (saddle point)
One or both zero	0	Cylindrical/conical (ridge, hollow, plane)

Example 6–11 Developable Surface

Show that an elliptic cone is a developable surface.

Rewriting Eq. (6–32) for a parametric elliptic cone in terms of u and w yields

$$Q(u,w) = [\, au\cos w \quad bu\sin w \quad cu \,]$$

The partial derivatives are

$$\begin{aligned}
Q_u &= [\, a\cos w \quad b\sin w \quad c \,] \\
Q_w &= [\, -au\sin w \quad bu\cos w \quad 0 \,] \\
Q_{uw} &= [\, -a\sin w \quad b\cos w \quad 0 \,] \\
Q_{uu} &= [\, 0 \quad 0 \quad 0 \,] \\
Q_{ww} &= [\, -au\cos w \quad -bu\sin w \quad 0 \,] \\
Q_u \times Q_w &= [\, -bcu\cos w \quad -acu\sin w \quad abu \,] \\
|Q_u \times Q_w|^2 &= (abu)^2\left\{\left(\frac{c}{a}\cos w\right)^2 + \left(\frac{c}{b}\sin w\right)^2 + 1\right\} \neq 0 \qquad u > 0
\end{aligned}$$

and

$$\begin{aligned}
A &= [\, -bcu\cos w \quad -acu\sin w \quad abu \,]\cdot[\, 0 \quad 0 \quad 0 \,] = 0 \\
B &= [\, -bcu\cos w \quad -acu\sin w \quad abu \,]\cdot[\, -a\sin w \quad b\cos w \quad 0 \,] \\
&= abcu\sin w\cos w - abcu\sin w\cos w = 0 \\
C &= [\, -bcu\cos w \quad -acu\sin w \quad abu \,]\cdot[\, -au\cos w \quad -bu\sin w \quad 0 \,] \\
&= abcu^2\cos^2 w + abcu^2\sin^2 w = abcu^2
\end{aligned}$$

Hence, using Eqs. (6–48)

$$K = \frac{AC - B^2}{|Q_u \times Q_w|^4} = \frac{(0)(abcu^2) - (0)}{|Q_u \times Q_w|^4} = 0$$

everywhere on the surface, and the surface is developable. Incidentally, note that although for $u = 0$, $|Q_u \times Q_w|^2 = 0$, use of L'Hôpital's rule shows that $K = 0/0 = 0$ at $u = 0$.

6–9 LINEAR COONS SURFACE

If the four boundary curves $P(u,0)$, $P(u,1)$, $P(0,w)$ and $P(1,w)$ are known, and a bilinear blending function is used for the interior of the surface patch, a linear Coons surface is obtained. At first glance it might be assumed that a simple sum of the singly ruled surfaces (Eqs. (6–43) and (6–44)) in the two directions